On the Analysis of Weighted Nonbinary Repeat Multiple-Accumulate Codes

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***Abstract*—In this paper, we consider weighted nonbinary repeat multiple-accumulate (WNRMA) code ensembles obtained from the serial concatenation of a nonbinary rate-$1/n$ repeat code and the cascade of $L \geq 1$ accumulators, where each encoder is followed by a nonbinary random weighter. The WNRMA codes are assumed to be iteratively decoded using the turbo principle with maximum *a posteriori* constituent decoders. We derive the exact weight enumerator of nonbinary accumulators and subsequently give the weight enumerators for WNRMA code ensembles. We formally prove that the symbol-wise minimum distance of WNRMA code ensembles asymptotically grows linearly with the block length when $L \geq 3$ and $n \geq 2$, and $L = 2$ and $n \geq 3$, for all powers of primes $q \geq 3$ considered, where q is the field size. Thus, WNRMA code ensembles are *asymptotically good* for these parameters. We also give iterative decoding thresholds, computed by an extrinsic information transfer chart analysis, on the $q$ary symmetric channel to show the convergence properties. Finally, we consider the binary image of WNRMA code ensembles and compare the asymptotic minimum distance growth rates with those of binary repeat multiple-accumulate code ensembles.**

I. Introduction

Weighted nonbinary repeat accumulate (WNRA) codes were introduced by Yang in [1] as the $q$ary generalization of the celebrated binary repeat accumulate (RA) codes. The encoder consists of a rate $R_{\rm rep} = 1/n$ nonbinary repeat code, a weighter, a random symbol interleaver, and an accumulator over a finite field GF(q) of size q. WNRA codes can be decoded iteratively using the turbo principle, and in [1] simulation results were presented that showed that these codes are superior to binary RA codes on the additive white Gaussian noise (AWGN) channel when the weighter is properly chosen. In a recent work [2], Kim *et al.* derived an approximate *input-output weight enumerator* (IOWE) for the nonbinary accumulator. Based on that, approximate upper bounds on the maximum-likelihood (ML) decoding threshold of WNRA codes with $q$ary orthogonal modulation and coherent detection over the AWGN channel were computed for different values of the repetition factor n and the field size q, showing that these codes perform close to capacity under ML decoding for large values of n and q.

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In [3], Pfister showed that the minimum distance ($d_{\min}$) of binary repeat multiple-accumulate (RMA) codes, built from the concatenation of a repeat code with two or more accumulators, increases as the number of accumulators increase. In particular, it was shown in [3] that there exists a sequence of RMA codes with $d_{\min}$ converging in the limit of infinitely many accumulators to the Gilbert-Varshamov bound (GVB). The stronger result that the typical $d_{\min}$ converges to the GVB was recently proved in [4]. Also, in [5], it was conjectured by Pfister that the $d_{\min}$ of RMA codes asymptotically grows linearly with the block length, and that the growth rate is given by the threshold where the asymptotic spectral shape function becomes positive. More recently, it was shown in [4, 6] that RMA code ensembles with two or more accumulators are indeed *asymptotically good*, in the sense that their $d_{\min}$ asymptotically grows linearly with the block length. A formal proof was given in [4], and a method for the calculation of a lower bound on the growth rate coefficient was given in [6].

In a recent paper [7], the authors considered weighted nonbinary repeat multiple-accumulate (WNRMA) code ensembles obtained from the serial concatenation of a nonbinary repeat code and the cascade of $L \geq 1$ accumulators, where each encoder is followed by a nonbinary weighter, as the $q$ary generalization of binary RMA codes [3–6, 8]. Building upon the approximate IOWE for nonbinary accumulators [2], it was shown numerically in [7] that the $d_{\min}$ of WNRMA code ensembles grows linearly with the block length, and the growth rates were estimated. However, no formal proof was provided in [7]. In this paper, we address this issue. We derive an *exact* expression for the IOWE of a nonbinary accumulator which allows us to derive an exact closed-form expression for the average *weight enumerator* (WE) of WNRMA code ensembles. We then analyze the asymptotic behavior of the average WE of WNRMA code ensembles, extending the asymptotic $d_{\min}$ analysis in [4, 6] for binary RMA code ensembles to WNRMA code ensembles. In particular, we prove that the $d_{\min}$ of WNRMA code ensembles asymptotically grows linearly with the block length when $L \geq 3$ and $n \geq 2$, and $L = 2$ and $n \geq 3$, for all powers of primes $q \geq 3$ considered. Hence, WNRMA code ensembles are asymptotically good for these parameters. The obtained growth rates are very close to the GVB for practical values of q. However, for large values of q, the growth rate coefficient decreases with q, and the gap to the GVB starts to increase. Furthermore, we consider extrinsic information transfer (EXIT) charts [9] to analyze the convergence properties of WNRMA codes on the $q$ary symmetric channel (QSC). Finally, we also consider the binary image of WNRMA codes. We give an expression for the average binary WE of nonbinary WNRMA code ensembles

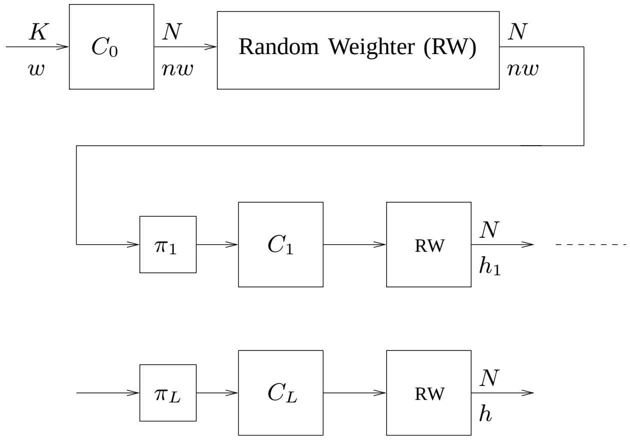

Fig. 1. Encoder structure for WNRMA codes.

and analyze its asymptotic behavior. We also compute the asymptotic $d_{\min}$ growth rates of the binary image of WNRMA code ensembles and compare them with those of binary RMA code ensembles. For given n, we show that the growth rate improves with the value of q for the considered values of q. Also, we compute ML decoding thresholds of the binary image of WNRMA code ensembles on the AWGN channel and show that these codes perform very close to capacity under ML decoding.

Nonbinary codes of low rate are potentially useful in image watermarking applications. See, for instance, [10] where low-rate nonbinary turbo codes were proposed for this application.

The remainder of the paper is organized as follows. In Section II, we describe the encoder structure of WNRMA codes. We also derive an exact expression for the IOWE of a nonbinary accumulator and a closed-form expression for the average WE of WNRMA code ensembles. In Section III, we analyze the asymptotic behavior of the average WE of WNRMA code ensembles and prove that its $d_{\min}$ grows linearly with the block length. Convergence properties under iterative decoding are studied in Section IV, where an EXIT chart analysis is performed. In Section V, we consider the binary image of WNRMA code ensembles and compare the $d_{\min}$ growth rates with those of binary RMA code ensembles. We also derive ML decoding thresholds for these ensembles. Finally, Section VI draws some conclusions.

II. Encoder Structure and Weight Enumerators

The encoder structure of WNRMA codes is depicted in Fig. 1. It is the serial concatenation of a rate $R_{\rm rep} = 1/n$ repetition code $C_{\rm rep}$, with the cascade of $L \geq 1$ identical rate-1, memory-one, $q$ary accumulators C_l, $l = 1, \ldots, L$, with generator polynomials $g(D) = 1/(1+D)$ over a finite field GF(q), through random interleavers $\pi_1, \ldots, \pi_L$. Each encoder is followed by a nonbinary weighter, which multiplies each symbol at its input by a nonzero $q$ary symbol. For analysis purposes we consider random weighters (RWs). We denote by C_0 the (nK, K) outer block code obtained by concatenating together K successive codewords of $C_{\rm rep}$. The overall nominal code rate (avoiding termination) is denoted by $R = K/N = 1/n$, where $N = nK$ is the output block length. In more detail, a length-K information sequence $\mathbf{u}_0 = (u_{0,1}, \ldots, u_{0,K})$ of $q$ary symbols $u_{0,i} \in \{0, 1, \ldots, q-1\}$ is encoded by a $q$ary repeat code. The output of the repeat code $\mathbf{x}_0 = (x_{0,1}, \ldots, x_{0,nK})$ is fed to a nonbinary weighter which multiplies each symbol $x_{0,i}$ by a nonzero $q$ary symbol. In [1], it was shown that a careful choice of the weighter can significantly improve performance. The resulting sequence is encoded by a chain of L nonbinary accumulators, preceded by interleavers $\pi_1, \ldots, \pi_L$. Furthermore, each accumulator is followed by a nonbinary RW.

A. Average WEs for WNRMA Code Ensembles

Let $\bar{a}^{\mathcal{C}}_{w,h}$ be the ensemble-average nonbinary IOWE of the code ensemble $\mathcal{C}$ with input and output block length K and N, respectively, denoting the average number of codewords of input Hamming weight w and output Hamming weight h over $\mathcal{C}$. Here, by Hamming weight, we mean the number of nonzero symbols in a codeword. For convenience, we may simply speak of weight. Also, denote by $\bar{a}^{\mathcal{C}}_h = \sum_{w=0}^{K} \bar{a}^{\mathcal{C}}_{w,h}$ the ensemble-average nonbinary WE of the code ensemble $\mathcal{C}$, giving the average number of codewords of weight h over $\mathcal{C}$. Throughout the paper we will simply speak of IOWE and WE, avoiding the term nonbinary, when the fact that they refer to nonbinary distributions is clear from the context.

Benedetto *et al.* introduced in [11] the concept of *uniform interleaver* to obtain average WEs for concatenated code ensembles from the WEs of the constituent encoders. Since we are dealing with nonbinary codes, we need to extend the approach from [11] to consider vector-WEs. In particular, consider the ensemble of serially concatenated codes (SCCs) obtained by connecting two nonbinary encoders $C_{\rm a}$ and $C_{\rm b}$ through a uniform interleaver. The ensemble-average IOWE of the serially concatenated code ensemble can be written as

$$\bar{a}^{\rm SCC}_{w,h} = \sum_{l} \sum_{\mathbf{l}: \sum_{i=1}^{q-1} l_i = l} \frac{a^{C_{\rm a}}_{w,\mathbf{l}} a^{C_{\rm b}}_{\mathbf{l},h}}{\binom{N}{l_1, l_2, \ldots, l_{q-1}}} \tag{1}$$

where

$$\binom{N}{l_1, l_2, \ldots, l_{q-1}} = \frac{N!}{l_1! \cdots l_{q-1}! (N - \sum_{i=1}^{q-1} l_i)!},$$

$\mathbf{l} = (l_1, l_2, \ldots, l_{q-1})$ is the weight vector with entries l_i giving the number of symbols i in a codeword $\mathbf{x}$, and $a^{C_{\rm a}}_{w,\mathbf{l}}$ is the vector-IOWE of encoder $C_{\rm a}$, giving the number of codewords of input weight w at the input of $C_{\rm a}$ and output vector-weight $\mathbf{l}$ at the output of $C_{\rm a}$, i.e., the codeword has l_1 1's, l_2 2's, and so on. Likewise, $a^{C_{\rm b}}_{\mathbf{l},h}$ is the vector-IOWE of encoder $C_{\rm b}$ giving the number of codewords of input vector-weight $\mathbf{l}$ and output weight h. In general, it is very difficult to compute the vector-IOWE of an encoder in closed-form. However, if encoder $C_{\rm a}$ is followed by a nonbinary RW, the following theorem holds.

Theorem 1: Let $\mathcal{C}$ be the ensemble of codes over GF(q) obtained by the serial concatenation of two nonbinary encoders $C_{\rm a}$ and $C_{\rm b}$ through a uniform interleaver. Furthermore, encoder $C_{\rm a}$ is followed by a nonbinary RW. Also, denote by $a^{C_{\rm a}}_{w,h}$ and $a^{C_{\rm b}}_{w,h}$ the IOWE of encoder $C_{\rm a}$ and encoder $C_{\rm b}$,

respectively. The ensemble-average IOWE of the ensemble $\mathcal{C}$ can be written as

$$\bar{a}^{\mathcal{C}}_{w,h} = \sum_{l} \frac{a^{C_{\rm a}}_{w,l} a^{C_{\rm b}}_{l,h}}{\binom{N}{l}(q-1)^l}. \tag{2}$$

Proof: Denote by $\mathcal{C}'_{\rm a}$ the ensemble obtained by joining together encoder $C_{\rm a}$ and the RW. Using the concept of uniform interleaver, the ensemble-average IOWE of the ensemble $\mathcal{C}$ can be written as (see (1))

$$\bar{a}^{\mathcal{C}}_{w,h} = \sum_{l} \sum_{\mathbf{l}:\sum_{i=1}^{q-1} l_i = l} \frac{\bar{a}^{\mathcal{C}'_{\rm a}}_{w,\mathbf{l}} a^{C_{\rm b}}_{\mathbf{l},h}}{\binom{N}{l_1,l_2,\ldots,l_{q-1}}} \tag{3}$$

where $\bar{a}^{\mathcal{C}'_{\rm a}}_{w,\mathbf{l}}$ is the average vector-IOWE of the ensemble of weighted codes $C_{\rm a}$, weighted through the RW.

The average vector-IOWE of the ensemble $\mathcal{C}'_{\rm a}$ can be written as a function of the vector-IOWEs of encoder $C_{\rm a}$ and of the RW as

$$\bar{a}^{\mathcal{C}'_{\rm a}}_{w,\mathbf{l}} = \sum_{\mathbf{d}} a^{C_{\rm a}}_{w,\mathbf{d}} a^{\rm RW}_{\mathbf{d},\mathbf{l}}. \tag{4}$$

Notice that the RW (over GF(q)) is such that the weight is preserved, i.e., $a^{\rm RW}_{\mathbf{d},\mathbf{l}}$ is nonzero if and only if $\sum_{i=1}^{q-1} d_i = \sum_{i=1}^{q-1} l_i$. Therefore, we can rewrite (4) as

$$\bar{a}^{\mathcal{C}'_{\rm a}}_{w,\mathbf{l}} = \sum_{\mathbf{d}:\sum_{i=1}^{q-1} d_i = \sum_{i=1}^{q-1} l_i} a^{C_{\rm a}}_{w,\mathbf{d}} a^{\rm RW}_{\mathbf{d},\mathbf{l}}.$$

Notice also that the following property holds for a nonbinary random (uniform) weighter:

$$a^{\rm RW}_{\mathbf{d},\mathbf{l}} = a^{\rm RW}_{\mathbf{d}',\mathbf{l}} \ \ \forall\, \mathbf{d},\mathbf{d}' \text{ such that } \sum_{i=1}^{q-1} d_i = \sum_{i=1}^{q-1} d'_i.$$

In other words, the vector-IOWE of the RW depends only on the weight $l = \sum_{i=1}^{q-1} l_i$, and we can write

$$a^{\rm RW}_{\mathbf{d},\mathbf{l}} = a^{\rm RW}_{l,\mathbf{l}} \ \ \forall\, \mathbf{d} \text{ such that } \sum_{i=1}^{q-1} d_i = l. \tag{5}$$

It is easy to verify that the vector-IOWE $a^{\rm RW}_{l,\mathbf{l}}$ is given by

$$a^{\rm RW}_{l,\mathbf{l}} = \frac{\binom{l}{l_1,l_2,\ldots,l_{q-1}}}{(q-1)^l}. \tag{6}$$

Finally, using (6), (5), (4), and the fact that

$$\sum_{\mathbf{d}:\sum_{i=1}^{q-1} d_i = l} a^{C_{\rm a}}_{w,\mathbf{d}} = a^{C_{\rm a}}_{w,l} \ \text{ and } \ \sum_{\mathbf{l}:\sum_{i=1}^{q-1} l_i = l} a^{C_{\rm b}}_{\mathbf{l},h} = a^{C_{\rm b}}_{l,h}$$

in (3), after some simple manipulations, we obtain (2), which completes the proof. ■

From Theorem 1 it follows that the ensemble-average IOWE of WNRMA code ensembles can be computed, when each constituent encoder is followed by a nonbinary RW, from the IOWEs of the component encoders, which are easier to compute in closed-form than the vector-IOWEs. Using Theorem 1 and the concept of uniform interleaver, the ensemble-average IOWE of a WNRMA code ensemble $\mathcal{C}_{\rm WNRMA}$ can be written as

$$\begin{aligned}\bar{a}^{\mathcal{C}_{\rm WNRMA}}_{w,h} &= \sum_{h_1=0}^{N} \cdots \sum_{h_{L-1}=0}^{N} \frac{a^{C_0}_{w,nw} a^{C_1}_{nw,h_1}}{\binom{N}{nw}(q-1)^{nw}} \\ &\quad \times \left[\prod_{l=2}^{L-1} \frac{a^{C_l}_{h_{l-1},h_l}}{\binom{N}{h_{l-1}}(q-1)^{h_{l-1}}} \right] \frac{a^{C_L}_{h_{L-1},h}}{\binom{N}{h_{L-1}}(q-1)^{h_{L-1}}} \\ &= \sum_{h_1=0}^{N} \cdots \sum_{h_{L-1}=0}^{N} \bar{a}^{\mathcal{C}_{\rm WNRMA}}_{w,h_1,\ldots,h_{L-1},h}\end{aligned} \tag{7}$$

where $\bar{a}^{\mathcal{C}_{\rm WNRMA}}_{w,h_1,\ldots,h_{L-1},h}$ is called the *conditional weight enumerator* (CWE) of $\mathcal{C}_{\rm WNRMA}$.

The evaluation of (7) requires the computation of the IOWEs of the constituent encoders, which is addressed below.

B. IOWEs for Memory-One Encoders and the Repetition Code

An approximated expression for the IOWE of a $q$ary accumulator was given in [2]. In this section, we derive the exact expression for the IOWE of a $q$ary accumulator.

Theorem 2: The IOWE for rate-1, memory-one, $q$ary convolutional encoders over GF(q) with generator polynomials $g(D) = 1/(1+D)$ and $g(D) = 1+D$ that are terminated to the zero state at the end of the trellis and with input and output block length N can be given in closed form as

$$\begin{aligned}a^{\frac{1}{1+D}}_{w,h} = a^{1+D}_{h,w} = &\sum_{k=\max(1,w-h)}^{\lfloor w/2 \rfloor} \binom{N-h}{k}\binom{h-1}{k-1}\binom{h-k}{w-2k} \\ &\times (q-1)^k (q-2)^{w-2k}\end{aligned} \tag{8}$$

for positive input weights w, where k is the number of error events. Also, $a^{\frac{1}{1+D}}_{0,0} = a^{1+D}_{0,0} = 1$.

Proof: Consider a nonbinary encoder C with input and output length N. Denote by an error event a path through the trellis which diverges from the all-zero state at depth $t_{\rm i}$ and merges again with the all-zero state at depth $t_{\rm f}$, where $t_{\rm f} > t_{\rm i}$. A nonzero codeword of input weight w and output weight h corresponds to the concatenation of k error events with a total input weight w and a total output weight h. Partition all the error events into equivalence classes based on their length (or, equivalently, based on their accumulated output weight). In particular, all error events within a specific class are required to have the same length. By considering only classes of events (i.e., we do not distinguish between error events within the same class), k error events with an accumulated output weight h can be concatenated (without overlapping) in

$$\binom{N-h}{k}\binom{h-1}{k-1}$$

different ways.

The next step is to consider all the error events within the same class. First, take a look at the structure of the error events. Notice that the first transition of an error event (the one diverging from the all-zero state) has always input weight one and output weight one, while the last transition of an error

$$\bar{a}^{\mathcal{C}_{\rm WNRMA}}_{w,h_1,\ldots,h_{L-1},h} = \frac{\binom{K}{w}(q-1)^w \prod_{l=1}^{L} \sum_{k_l=\max(1,h_{l-1}-h_l)}^{\lfloor h_{l-1}/2 \rfloor} \binom{N-h_l}{k_l}\binom{h_l-1}{k_l-1}\binom{h_l-k_l}{h_{l-1}-2k_l}(q-1)^{k_l}(q-2)^{h_{l-1}-2k_l}}{\prod_{l=1}^{L}\binom{N}{h_{l-1}}(q-1)^{h_{l-1}}}$$
$$= \sum_{k_1=\max(1,h_0-h_1)}^{\lfloor h_0/2 \rfloor} \sum_{k_2=\max(1,h_1-h_2)}^{\lfloor h_1/2 \rfloor} \cdots \sum_{k_L=\max(1,h_{L-1}-h_L)}^{\lfloor h_{L-1}/2 \rfloor} \bar{a}^{\mathcal{C}_{\rm WNRMA}}_{w,h_1,\ldots,h_{L-1},k_1,\ldots,k_L,h} \tag{10}$$

event (the one merging with the all-zero state) has always input weight one and output weight zero. Thus, the total input weight accumulated at the boundaries (first and last transition) of the error events is $2k$, while the total output weight accumulated at the boundaries of the error events is k. Now, each error event has $q-1$ possibilities for the first transition (since the edge from the all-zero state to the all-zero state is not allowed), therefore, overall, we have $(q-1)^k$ possibilities. On the other hand, there is only a single possibility for the last transition (the one merging with the all-zero state). Finally, we must distribute the remaining input weight, $w-2k$, in the $h-k$ remaining transitions (i.e., excluding the boundaries) of the error events, resulting in

$$\binom{h-k}{w-2k}$$

possible distributions for the remaining input weight. Furthermore, for each of the nonzero input weight transitions, we have $q-2$ additional possibilities (we must exclude the edge of input weight zero and the edges merging with the all-zero state), resulting in $(q-2)^{w-2k}$ possibilities in total. Thus, overall there are

$$\binom{N-h}{k}\binom{h-1}{k-1}(q-1)^k\binom{h-k}{w-2k}(q-2)^{w-2k}$$

codewords of input weight w and output weight h resulting from the concatenation of k error events. The result for the encoder $g(D)=1/(1+D)$ in (8) follows by summing over all possible values of k. The IOWE for the feedforward encoder with generator polynomial $g(D)=1+D$ is obtained in a similar manner. ■

Notice that the formula in (8) generalizes the closed-form expression for the IOWE for rate-1, memory-one, binary convolutional encoders from [12] to the $q$ary case.

Theorem 3: The IOWE for the (nK,K) $q$ary repetition code C_0 with input block length K can be given in closed form as

$$a^{C_0}_{w,nw} = \binom{K}{w}(q-1)^w. \tag{9}$$

Proof: The number of binary vectors of length K and weight w is $\binom{K}{w}$, and the result follows by multiplying this number by w times the number of nonzero elements from GF(q). ■

Using (8) and (9) in (7), we get the expression (10) at the top of the page for the CWE (with $w>0$) of WNRMA code ensembles, where for conciseness $h_0=nw$ and $h_L=h$.

III. Asymptotic Analysis of the Minimum Distance

With regard to (10) at the top of the page, without loss of generality we can write

$$w=\alpha N^a, \qquad h_i=\beta_i N^{b_i}, i=1,\ldots,L-1,$$
$$h=\rho N^c, \qquad k_i=\gamma_i N^{d_i}, i=1,\ldots,L$$

where $0\le a\le b_1\le b_2\le\cdots\le b_{L-1}\le c\le 1$, $0\le d_1\le a\le 1$, and $0\le d_i\le b_{i-1}\le 1$, $i=2,\ldots,L$. These inequalities can be derived from the binomial coefficients in the expression in (10) combined with the fact that for a binomial coefficient $\binom{n}{k}$, $n\ge k\ge 0$. Also, $\alpha,\beta_1,\ldots,\beta_{L-1},\gamma_1,\ldots,\gamma_L$, and ρ are positive constants. We must consider two cases: 1) at least one of the quantities $w,h_1,\ldots,h_{L-1},k_1,\ldots,k_L$, or h is of order $o(N)$, and 2) all quantities $w,h_1,\ldots,h_{L-1},k_1,\ldots,k_L$, and h can be expressed as fractions of the block length N, i.e., $a=b_1=\cdots=b_{L-1}=d_1=\cdots=d_L=c=1$. The following lemma addresses the first case for weighted nonbinary repeat double-accumulate (WNRAA) code ensembles.

Lemma 1: In the ensemble of WNRAA codes with block length N and $n\ge 3$, in the case where at least one of the quantities w, h_1, k_1, k_2, or h is of order $o(N)$, $N^5\bar{a}^{\mathcal{C}_{\rm WNRAA}}_{w,h_1,k_1,k_2,h}\longrightarrow 0$ as $N\longrightarrow\infty$ for all positive values of h.

Proof: The expression in (10) is very similar to the expression for the conditional support size enumerating function of RMA code ensembles [8, Eq. (10)]. In particular, the binomial coefficients in (10) are identical to those of [8, Eq. (10)]. The only difference is that (10) has some extra terms in the form of powers of $q-1$ and $q-2$. Therefore, the proof of [8, Lemma 3] applies, with some modifications, also here. ■

Lemma 1 can be generalized to the case of WNRMA code ensembles with $L\ge 3$. The proof is omitted for brevity. As a consequence of Lemma 1, we can assume that w, $h_1,\ldots,h_{L-1},k_1,\ldots,k_L$, and h are all linear in the block length: The average number of codewords of weight at most $\hbar$, for some $\hbar$, of WNRMA code ensembles is upper-bounded by

$$N^{2L+1}\max_{w,h_1,\ldots,h_{L-1},k_1,\ldots,k_L,h\le\hbar}\bar{a}^{\mathcal{C}_{\rm WNRMA}}_{w,h_1,\ldots,h_{L-1},k_1,\ldots,k_L,h}$$

which from Lemma 1 tends to zero as N tends to infinity if at least one of the quantities is of order $o(N)$. Thus, the average number of codewords of sublinear weight of at most $\hbar$ tends to zero as N tends to infinity.

We now address the second case by analyzing the asymptotic spectral shape function. The asymptotic spectral shape function is defined as [13]

$$r(\rho)=\limsup_{N\longrightarrow\infty}\frac{1}{N}\ln\bar{a}^{\mathcal{C}}_{\lfloor\rho N\rfloor}$$

where $\sup(\cdot)$ denotes the supremum of its argument, $\rho = \frac{h}{N}$ is the normalized output weight, and N is the code block length. If there exists some abscissa $\rho_0 > 0$ such that $\sup_{\rho \leq \rho^*} r(\rho) < 0 \quad \forall \rho^* < \rho_0$, and $r(\rho) > 0$ for some $\rho > \rho_0$, then it can be shown that, with high probability, the $d_{\min}$ of most codes in the ensemble grows linearly with the block length N, with growth rate coefficient of at least ρ_0. On the other hand, if $r(\rho)$ is strictly zero in the range $(0, \rho_0)$, it cannot be proved directly whether the $d_{\min}$ grows linearly with the block length or not. In [4], it was shown that the asymptotic spectral shape function of RMA codes exhibits this behavior, i.e., it is zero in the range $(0, \rho_0)$ and positive for some $\rho > \rho_0$. By combining the asymptotic spectral shapes with the use of bounding techniques, it was proved in [4, Theorem 6] that the $d_{\min}$ of RMA code ensembles indeed grows linearly with the block length with growth rate coefficient of at least ρ_0.

We remark that in the rest of the paper, with a slight abuse of language, we sometimes refer to ρ_0 as the exact value of the asymptotic growth rate coefficient. However, strictly speaking, ρ_0 is only a lower bound on it.

Now, by using Stirling's approximation for the binomial coefficient $\binom{n}{k} \sim \mathrm{e}^{n\mathbb{H}(k/n)}$ for $n \to \infty$ and k/n constant, where $\mathbb{H}(\cdot)$ is the binary entropy function with natural logarithms, and the fact that $w, h_1, \ldots, h_{L-1}, k_1, \ldots, k_L$, and h can all be assumed to be of the same order as N (due to Lemma 1, generalized to the general case), $\bar{a}^{\mathcal{C}_{\mathrm{WNRMA}}}_{w,h_1,\ldots,h_{L-1},h}$ can be written as

$$\bar{a}^{\mathcal{C}_{\mathrm{WNRMA}}}_{w,h_1,\ldots,h_{L-1},h} = \sum_{k_1,\ldots,k_L} \exp\left\{ f(\alpha, \beta_1, \ldots, \beta_{L-1}, \gamma_1, \ldots, \gamma_L, \rho)\, N + o(N) \right\}$$

when $N \longrightarrow \infty$, where $\alpha = \frac{w}{K}$ is the normalized input weight, $\beta_l = \frac{h_l}{N}$ is the normalized output weight of code C_l, $\gamma_l = \frac{k_l}{N}$, and the function $f(\cdot)$ is given by

$$\begin{aligned} f(&\beta_0, \beta_1, \ldots, \beta_{L-1}, \gamma_1, \ldots, \gamma_L, \rho) \\ &= \frac{\mathbb{H}(\beta_0)}{n} - \sum_{l=1}^{L} \mathbb{H}(\beta_{l-1}) + \sum_{l=1}^{L} (1-\beta_l)\mathbb{H}\left(\frac{\gamma_l}{1-\beta_l}\right) \\ &\quad + \sum_{l=1}^{L} \beta_l \mathbb{H}\left(\frac{\gamma_l}{\beta_l}\right) + \sum_{l=1}^{L} (\beta_l - \gamma_l)\mathbb{H}\left(\frac{\beta_{l-1} - 2\gamma_l}{\beta_l - \gamma_l}\right) \\ &\quad + \ln(q-1) \sum_{l=1}^{L} (\gamma_l - \beta_{l-1}) \\ &\quad + \ln(q-2) \sum_{l=1}^{L} (\beta_{l-1} - 2\gamma_l) + \frac{\beta_0 \ln(q-1)}{n} \end{aligned} \tag{11}$$

where for conciseness we defined $\beta_0 = \alpha$ and $\beta_L = \rho$. Finally, the asymptotic spectral shape function for WNRMA code ensembles can be written as

$$r^{\mathcal{C}_{\mathrm{WNRMA}}}(\rho) = \sup_{\substack{0 \leq \beta_{l-1} \leq 1 \\ \max(0, \beta_{l-1} - \beta_l) \leq \gamma_l \leq \\ \min(\beta_l, 1-\beta_l, \beta_{l-1}/2) \\ l=1,\ldots,L}} f(\beta_0, \beta_1, \ldots, \beta_{L-1}, \gamma_1, \ldots, \gamma_L, \rho). \tag{12}$$

Note that the objective function in (12), defined in (11), can be rewritten into [7, Eq. (6)], since

$$\begin{aligned} &\sum_{l=1}^{L} \beta_l \mathbb{H}\left(\frac{\gamma_l}{\beta_l}\right) + \sum_{l=1}^{L} (\beta_l - \gamma_l)\mathbb{H}\left(\frac{\beta_{l-1} - 2\gamma_l}{\beta_l - \gamma_l}\right) \\ &= \sum_{l=1}^{L} \beta_l \mathbb{H}\left(\frac{\beta_{l-1} - \gamma_l}{\beta_l}\right) + \sum_{l=1}^{L} (\beta_{l-1} - \gamma_l)\mathbb{H}\left(\frac{\gamma_l}{\beta_{l-1} - \gamma_l}\right). \end{aligned}$$

Thus, the approximate asymptotic spectral shape function given in [7, Eq. (7)] is indeed exact. Therefore, the growth rate coefficients computed in this section coincide with those in [7]. However, for finite block lengths, the IOWE of a nonbinary accumulator as given by Theorem 1 in [7] using the approximation for $p(k)$ given in [7, Eq. (3)] (which is taken from [2]) is not exact.

From (11) and (12) it can easily be verified that the asymptotic spectral shape function of WNRMA code ensembles satisfies the recursive relation

$$r^{\mathcal{C}_{\mathrm{WNRMA}(l)}}(\rho) = \sup_{0 \leq u \leq 1} \left[r^{\mathcal{C}_{\mathrm{WNRMA}(l-1)}}(u) + \psi(u, \rho) \right]$$

where $r^{\mathcal{C}_{\mathrm{WNRMA}(l)}}$, $l > 0$, is the asymptotic spectral shape function with l accumulators, $r^{\mathcal{C}_{\mathrm{WNRMA}(0)}}(\rho) = \frac{1}{n}(H(\rho) + \rho \ln(q-1))$ is the asymptotic spectral shape function of a repeat code, and

$$\begin{aligned} \psi(u, \rho) = \sup_{\substack{\max(0, u-\rho) \leq \gamma \leq \\ \min(\rho, 1-\rho, u/2)}} &\left[-\mathbb{H}(u) + \rho \mathbb{H}\left(\frac{\gamma}{\rho}\right) \right. \\ &+ (1-\rho)\mathbb{H}\left(\frac{\gamma}{1-\rho}\right) + (\rho - \gamma)\mathbb{H}\left(\frac{u - 2\gamma}{\rho - \gamma}\right) \\ &\left. + (\gamma - u)\ln(q-1) + (u - 2\gamma)\ln(q-2) \right]. \end{aligned} \tag{13}$$

Lemma 2: The asymptotic spectral shape function of the WNRMA code ensemble is nonnegative, i.e.,

$$r^{\mathcal{C}_{\mathrm{WNRMA}(l)}}(\rho) \geq 0, \ \forall \rho \in [0, 1].$$

Proof: We have $r^{\mathcal{C}_{\mathrm{WNRMA}(1)}}(\rho) \geq \psi(0, \rho) + H(0)/n = 0$. The general case can be proved by induction on l. ■

To analyze the asymptotic $d_{\min}$ behavior of WNRMA code ensembles, we must solve the optimization problem in (11)-(12). An efficient algorithm to solve this problem is given in Appendix A. The numerical evaluation of (11)-(12) is shown in Figs. 4 and 5 for WNRAA and weighted nonbinary repeat triple-accumulate (WNRAAA) code ensembles, respectively, with $n = 3$ and $q = 4, 8, 16$, and 32. The asymptotic spectral shape function is zero in the range $(0, \rho_0)$ and positive for some $\rho > \rho_0$. In this case, we cannot conclude directly whether the $d_{\min}$ asymptotically grows linearly with the block length or not. However, we can prove the following theorem.

Theorem 4: Define $\rho_0 = \max\{\rho^* \in [0, (q-1)/q) : r^{\mathcal{C}_{\mathrm{WNRMA}}}(\rho) = 0 \ \forall \rho \leq \rho^*\}$. Then $\forall \rho^* > 0$

$$\lim_{N \longrightarrow \infty} \Pr\left(d_{\min} \leq (\rho_0 - \rho^*)N\right) = 0$$

when $L \geq 3$ and $n \geq 2$, and $L = 2$ and $n \geq 3$, for all powers of primes $q \geq 3$. Thus, if $\rho_0 > 0$ and $r^{\mathcal{C}_{\mathrm{WNRMA}}}(\rho) \geq 0 \ \forall \rho$ (see Lemma 2), then almost all codes in the ensemble have

asymptotic minimum distance growing linearly with N with growth rate coefficient of at least ρ_0.

Proof: See Appendix B. ∎

We can now prove the following theorem.

Theorem 5: The typical $d_{\min}$ of WNRMA code ensembles when $L \geq 3$ and $n \geq 2$, and $L = 2$ and $n \geq 3$, for all powers of primes $3 \leq q \leq 2^{25}$, grows linearly with the block length.

Proof: The result follows from Theorem 4 by showing that ρ_0 (as defined in Theorem 4) is strictly positive when $L \geq 3$ and $n \geq 2$, and $L = 2$ and $n \geq 3$, for all powers of primes $3 \leq q \leq 2^{25}$.

It follows directly from the definition of the objective function in (11) that the asymptotic spectral shape function is nonincreasing in n, and thus ρ_0 is nondecreasing in n.

Furthermore, note that if we serially concatenate any nonbinary encoder whose $d_{\min}$ grows linearly with the block length with growth rate coefficient of at least ρ_0 with a nonbinary accumulate code followed by a uniform weighter through a uniform interleaver, the resulting concatenated code ensemble will exhibit a $d_{\min}$ growing linearly with the block length with growth rate coefficient of at least $\lceil \rho_0/2 \rceil$. This follows from the fact that the output weight h of a nonbinary accumulate code is lower bounded by $\lceil \frac{w}{2} \rceil$, where w is the nonbinary input weight. This follows directly from the binomial coefficient $\binom{h-k}{w-2k}$ in (8), since it implies that $h - k \geq w - 2k$, from which it follows that $w \leq h + k \leq h + \lfloor w/2 \rfloor$, which implies that $h \geq \lceil \frac{w}{2} \rceil$.

Thus, increasing n or L does not change the asymptotic $d_{\min}$ linear growth property. The final part of the proof considers the last *dimension*, i.e., what happens when q increases.

By numerically solving the optimization problem in (12), we find that $\rho_0 = 0.1966$ for $q = 3$, $n = 3$, and $L = 2$, and $\rho_0 = 0.1519$ for $q = 3$, $n = 2$, and $L = 3$. Furthermore, Figs. 2 and 3 show the value of ρ_0 (computed numerically by solving the optimization problem in (12) as function of the field size q for $q = 3$ and $q = 2^l$, $2 \leq l \leq 25$, when $n = 3$ and $L = 2$, and $n = 2$ and $L = 3$, respectively. From the figures, we observe that ρ_0 is strictly positive for $q \leq 2^{25}$ in both cases, i.e., for both $n = 3$ and $L = 2$, and $n = 2$ and $L = 3$, which concludes the proof. ∎

We remark that we have limited the value of q to 2^{25}, which is much higher than any value used in practice. However, from Figs. 2 and 3, we observe that the result of Theorem 5 will also hold for larger values of q.

The exact values of ρ_0 are given in Table I for several values of the repetition factor n and the field size q for WNRAA codes. For comparison, we have also tabulated the asymptotic $d_{\min}$ growth rate coefficient from the asymptotic GVB for nonbinary codes computed from

$$R \geq \begin{cases} 1 - \mathbb{H}_q(\rho_{\min}) - \rho_{\min} \log_q(q-1), & \text{if } \rho_{\min} \leq \frac{q-1}{q} \\ 0, & \text{otherwise} \end{cases}$$

where $\rho_{\min}$ is the normalized $d_{\min}$, R is the asymptotic rate, and $\mathbb{H}_q(\cdot)$ is the binary entropy function with base-q logarithms. We observe that the gap to the GVB decreases with increasing values of n for a fixed value of q. For a fixed value of n, the growth rate coefficient increases with increasing

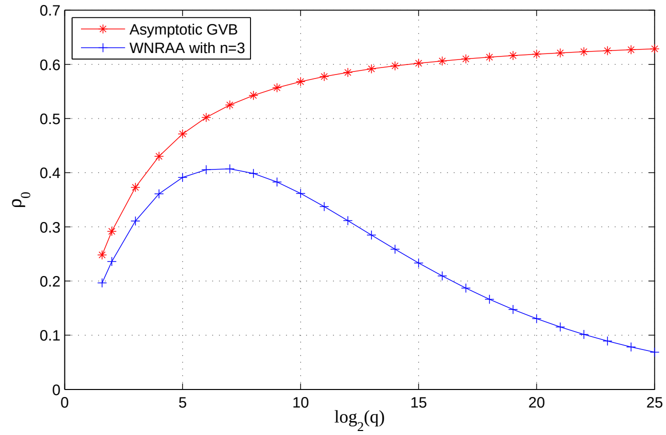

Fig. 2. The value of ρ_0 versus the field size q for $q = 3$ and $q = 2^l$, $2 \leq l \leq 25$, when $n = 3$ and $L = 2$.

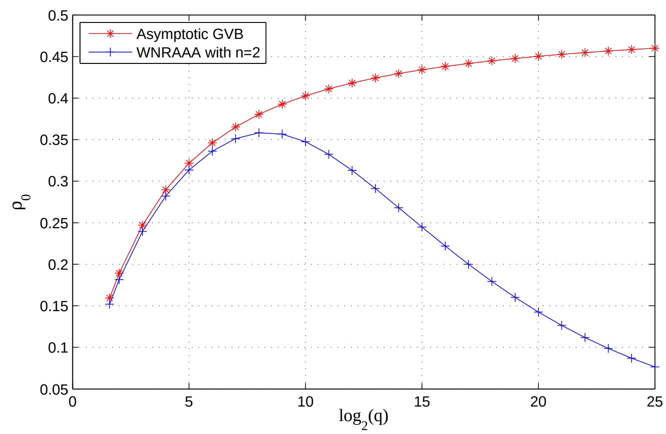

Fig. 3. The value of ρ_0 versus the field size q for $q = 3$ and $q = 2^l$, $2 \leq l \leq 25$, when $n = 2$ and $L = 3$.

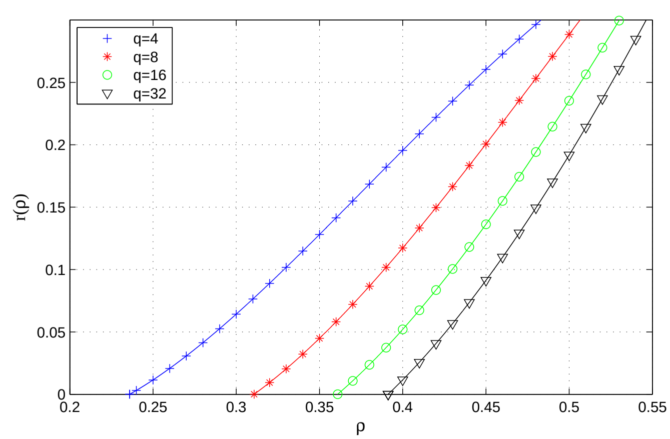

Fig. 4. Asymptotic spectral shape function of WNRAA codes with $n = 3$.

values of q, while the gap to the GVB stays approximately constant. However, as can be seen from Figs. 2 and 3, this behavior only holds for small values of q. In fact, the asymptotic growth rate coefficient increases with the field size q up to some value, and then it decreases again, after which the gap to the GVB also increases. This is also consistent with the behavior observed for nonbinary low-density parity-check

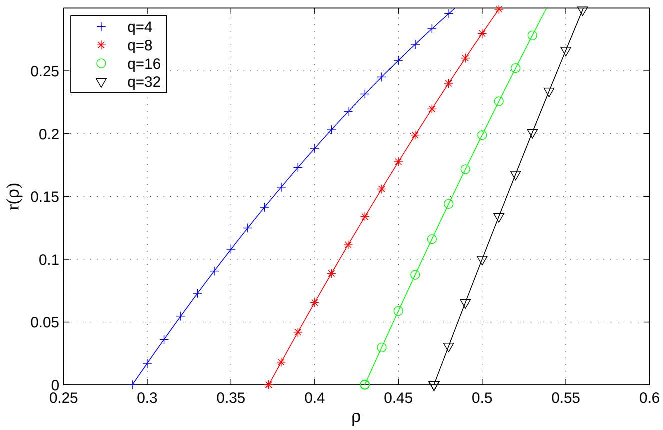

Fig. 5. Asymptotic spectral shape function of WNRAAA codes with $n = 3$.

TABLE I
GROWTH RATE COEFFICIENT ρ_0 OF WNRAA CODES FOR DIFFERENT VALUES OF THE REPETITION FACTOR n AND THE FIELD SIZE q. THE CORRESPONDING GROWTH RATES FROM THE ASYMPTOTIC NONBINARY GVB ARE GIVEN IN THE PARENTHESES.

	$q=4$	$q=8$	$q=16$	$q=32$
$n=3$	0.2360 (0.2917)	0.3107 (0.3730)	0.3609 (0.4302)	0.3912 (0.4715)
$n=5$	0.3820 (0.3977)	0.4840 (0.4987)	0.5518 (0.5664)	0.5967 (0.6131)
$n=10$	0.5026 (0.5048)	0.6192 (0.6207)	0.6930 (0.6940)	0.7413 (0.7421)

TABLE II
GROWTH RATE COEFFICIENT ρ_0 OF WNRAAA CODES FOR DIFFERENT VALUES OF THE REPETITION FACTOR n AND THE FIELD SIZE q. THE CORRESPONDING GROWTH RATES FROM THE ASYMPTOTIC NONBINARY GVB ARE GIVEN IN THE PARENTHESES.

	$q=4$	$q=8$	$q=16$	$q=32$
$n=3$	0.2911 (0.2917)	0.3725 (0.3730)	0.4299 (0.4302)	0.4712 (0.4715)
$n=5$	0.3977 (0.3977)	0.4987 (0.4987)	0.5664 (0.5664)	0.6131 (0.6131)
$n=10$	0.5048 (0.5048)	0.6207 (0.6207)	0.6940 (0.6940)	0.7421 (0.7421)

codes in [14]. The values of ρ_0 for WNRAAA code ensembles are given in Table II for selected values of n and q. The growth rate coefficients are very close to the GVB for WNRAA code ensembles with $n = 5$ and $n = 10$ and for WNRAAA code ensembles, for the considered values of q. For WNRAAA code ensembles with $n = 5$ and $n = 10$ the growth rates coincide with the GVB, for the considered values of q.

IV. EXIT CHART ANALYSIS

In this section, we analyze the convergence properties of iterative decoding of WNRMA codes on the QSC using the turbo principle by means of an EXIT chart analysis [9].

The QSC is characterized by a single parameter p, which is the error probability of the channel. The QSC with error probability p takes a $q$ary symbol at the input and outputs either the unchanged input symbol, with probability $1-p$, or any of the other $q-1$ symbols, with equal probability, i.e., with probability $p/(q-1)$. The capacity C (in bits per channel use) of the QSC with error probability p, assuming that $q = 2^m$, for some positive integer m, is given by [15]

$$C = m - \mathbb{H}_2(p) - p\log_2(q-1). \qquad (14)$$

Asymptotically, the normalized capacity C/m approaches $1-p$ as m tends to infinity, which is the capacity of a binary erasure channel with erasure probability p.

TABLE III
CONVERGENCE THRESHOLDS FOR WNRAA CODE ENSEMBLES FOR DIFFERENT VALUES OF THE REPETITION FACTOR n AND THE FIELD SIZE q ON THE QSC. THE CORRESPONDING CAPACITY VALUES (COMPUTED FROM (14) WITH $C/\log_2(q) = 1/n$) ARE GIVEN IN THE PARENTHESES.

	$q=4$	$q=8$	$q=16$	$q=32$
$n=3$	0.228 (0.292)	0.290 (0.373)	0.335 (0.430)	0.374 (0.471)
$n=5$	0.263 (0.398)	0.333 (0.499)	0.382 (0.566)	0.412 (0.613)
$n=10$	0.306 (0.505)	0.379 (0.621)	0.424 (0.694)	0.459 (0.742)

The EXIT chart for a WNRMA code ensemble can be computed by properly combining the EXIT functions of the L constituent encoders $C_0, \ldots, C_{L-1}$ (see Fig. 1) into a single EXIT function and then plot together in a two-dimensional chart this EXIT function and the EXIT function of encoder C_L. Note that each encoder is followed by a nonbinary RW. However, for notational simplicity, we assume that the RWs are included in $C_0, \ldots, C_{L-1}$, i.e., when we speak about the EXIT function of C_l, we are referring to the EXIT function of encoder C_l followed by a RW. Let $I_{e,\mathbf{u}_l}^{C_l}$ and $I_{e,\mathbf{x}_l}^{C_l}$ denote the extrinsic mutual information (MI) generated by decoder C_l^{-1} on input word $\mathbf{u}_l$ at the input of encoder C_l and on codeword $\mathbf{x}_l$ at the output of C_l, respectively. Likewise, we define the *a priori* MIs by $I_{a,\mathbf{u}_l}^{C_l}$ and $I_{a,\mathbf{x}_l}^{C_l}$. Consider the WNRMA encoder with encoders $C_0, \ldots, C_{L-1}$ as a single encoder and denote it by C_O. We can compute the EXIT functions

$$I_{e,\mathbf{x}_{L-1}}^{C_O} = T^{C_O}(I_{a,\mathbf{x}_{L-1}}^{C_O}) \text{ and } I_{e,\mathbf{u}_L}^{C_L} = T^{C_L}(I_{a,\mathbf{u}_L}^{C_L}, p) \qquad (15)$$

for encoders C_O and C_L, respectively, where $I_{a,\mathbf{x}_{L-1}}^{C_O} = I_{e,\mathbf{u}_L}^{C_L}$ and $I_{a,\mathbf{u}_L}^{C_L} = I_{e,\mathbf{x}_{L-1}}^{C_O}$. Note that $I_{e,\mathbf{x}_{L-1}}^{C_O}$ does not depend on the channel while $I_{e,\mathbf{u}_L}^{C_L}$ does, since C_L is connected to the channel.

The convergence behavior of WNRMA codes can now be tracked by displaying in a single plot the two EXIT curves in (15). The iterative decoding of WNRMA codes processes extrinsic information at the symbol level. Therefore, nonbinary EXIT chart analysis is required. To compute the EXIT functions in (15) we use the method proposed in [16] for turbo codes and serially concatenated codes, generalized to multiple serially concatenated codes.

The convergence thresholds for WNRAA code ensembles predicted by the EXIT chart analysis (the maximum value of p such that a tunnel between the two EXIT curves in (15) is observed) are given in Table III for several values of the repetition factor n and the field size q on a QSC. For comparison purposes, we also report in Table III the corresponding capacity values computed from (14). From Table III it can be observed that for a given n the gap to capacity is similar for different values of q. On the other hand, given q, the iterative thresholds of WNRAA code ensembles are further away from capacity for increasing values of n.

V. BINARY IMAGE OF WNRMA CODE ENSEMBLES

In this section, we consider the binary image of WNRMA code ensembles. We derive the average binary WE over the ensemble of binary images of WNRMA code ensembles, where each nonzero symbol from GF(q) is mapped uniformly at random to nonzero binary vectors. We then compute the $d_{\min}$ asymptotic growth rates and upper bounds on the ML

thresholds over an AWGN channel and compare them with those of binary RMA code ensembles.

Let C be a code of length N over GF(q), where $q = 2^m$, and let a_h^C be its nonbinary WE. Denote by $\bar{a}_d^{C,\mathrm{b}}$ the average binary WE over the ensemble of binary images of C, where each nonzero symbol from GF(q) is mapped uniformly at random to nonzero binary vectors of length m, giving the number of codewords of binary weight d. In the following, we will refer to this WE as the average binary WE of code C. Also, denote as before by $\rho = \frac{h}{N}$ the normalized nonbinary weight. Likewise, we denote by $\delta = \frac{d}{Nm}$ the normalized binary weight. The average binary WE of code C can be obtained from the nonbinary WE of the code as [17]

$$\bar{a}_{\lfloor \delta Nm \rfloor}^{C,\mathrm{b}} = \sum_{i=\lfloor \delta N \rfloor}^{\min(N,\lfloor \delta Nm \rfloor)} \frac{a_i^C}{(2^m-1)^i} \operatorname{coeff}\left(((1+x)^m - 1)^i , x^{\lfloor \delta Nm \rfloor} \right) \tag{16}$$

where $\operatorname{coeff}(p(x), x^i)$ is a shorthand notation for the coefficient of the monomial x^i (the second argument) in the polynomial $p(x)$ (the first argument).

The binary asymptotic spectral shape function of the WNRMA code ensemble is defined as [13]

$$r_{\mathrm{b}}^{\mathcal{C}_{\mathrm{WNRMA}}}(\delta) = \limsup_{N \longrightarrow \infty} \frac{1}{Nm} \ln \bar{a}_{\lfloor \delta Nm \rfloor}^{\mathcal{C}_{\mathrm{WNRMA},\mathrm{b}}} \tag{17}$$

where $\bar{a}_i^{\mathcal{C}_{\mathrm{WNRMA},\mathrm{b}}}$ denotes the average binary WE of the overall ensemble consisting of all possible binary images (obtained by mapping nonzero symbols from GF(q) to nonzero binary vectors of length m) of all codes in the WNRMA code ensemble.

We will make use of the following corollary.

Corollary 1 ([18], Corollary 16 with $d = 1$):

$$\lim_{N\to\infty} \frac{1}{N} \ln \operatorname{coeff}(p(x)^N, x^{N\xi}) = \ln p(\tilde{x}) - \xi \ln \tilde{x}$$

where $p(x)$ is a polynomial in x and $\tilde{x}$ is the smallest positive solution of

$$\left. \frac{\partial \ln p(\mathrm{e}^s)}{\partial s} \right|_{s=\ln x} = \xi.$$

Using (16) and Corollary 1 in (17), the binary asymptotic spectral shape function of the WNRMA code ensemble can be written as

$$\begin{aligned} r_{\mathrm{b}}^{\mathcal{C}_{\mathrm{WNRMA}}}(\delta) &= \frac{1}{m} \sup_{\delta \le \rho \le \min(1, m\delta)} \Big(r^{\mathcal{C}_{\mathrm{WNRMA}}}(\rho) - \rho \ln(2^m - 1) \\ &\quad + \rho \ln\left((1+\tilde{x})^m - 1\right) - m\delta \ln \tilde{x} \Big) \end{aligned} \tag{18}$$

where $\tilde{x}$ is the smallest positive solution to the polynomial equation

$$\rho x (1+x)^{m-1} = \delta((1+x)^m - 1)$$

which simplifies, using the binomial theorem, to

$$\sum_{j=1}^{m} \left(1 - \frac{m\delta}{j\rho}\right) \binom{m-1}{j-1} x^j = 0.$$

TABLE IV

BINARY IMAGE GROWTH RATE COEFFICIENT δ_0 OF WNRAA CODES FOR DIFFERENT VALUES OF THE REPETITION FACTOR n AND THE FIELD SIZE q. THE CORRESPONDING GROWTH RATES FROM THE ASYMPTOTIC BINARY GVB ARE GIVEN IN THE LAST COLUMN.

	$q=2$	$q=4$	$q=8$	$q=16$	$q=32$	GVB
$n=3$	0.1323 [4, 6]	0.1496	0.1608	0.1675	0.1712	0.1740
$n=5$	0.2286 [4, 6]	0.2380	0.2416	0.2427	0.2429	0.2430
$n=10$	0.3133 [4]	0.3155	0.3159	0.3160	0.3160	0.3160

TABLE V

BINARY IMAGE GROWTH RATE COEFFICIENT δ_0 OF WNRAAA CODES FOR DIFFERENT VALUES OF THE REPETITION FACTOR n AND THE FIELD SIZE q. THE CORRESPONDING GROWTH RATES FROM THE ASYMPTOTIC BINARY GVB ARE GIVEN IN THE LAST COLUMN.

	$q=2$	$q=4$	$q=8$	$q=16$	$q=32$	GVB
$n=3$	0.1731 [4, 6]	0.1738	0.1739	0.1739	0.1740	0.1740
$n=5$	0.2430 [4, 6]	0.2430	0.2430	0.2430	0.2430	0.2430
$n=10$	0.3160 [4]	0.3160	0.3160	0.3160	0.3160	0.3160

To analyze the asymptotic behavior of the binary $d_{\min}$ of WNRMA code ensembles, we must solve the optimization problem in (18), similarly to the nonbinary case. Notice that since the nonbinary $d_{\min}$ of WNRMA code ensembles grows linearly with the block length, see Theorem 5, it follows that the $d_{\min}$ of its binary image also grows linearly with the block length. In Table IV, we give the binary $d_{\min}$ growth rate coefficient δ_0 of WNRAA code ensembles for several values of n and q. As a comparison, we also report the values for the binary RAA code ensemble ($q = 2$). It is observed that WNRAA code ensembles achieve higher growth rates than RAA code ensembles. For $q = 32$ the growth rates are very close to the GVB. The asymptotic binary $d_{\min}$ growth rates for WNRAAA and RAAA code ensembles are reported in Table V.

A. Threshold Under ML Decoding

The asymptotic spectral shape function of a code ensemble can also be used to derive a threshold under ML decoding. An upper bound on the ML decoding threshold of a code ensemble on the AWGN channel, due to Divsalar [19], is given by

$$\left(\frac{E_b}{N_0}\right)_{\mathrm{ML,threshold}} \le \frac{1}{R} \cdot \max_{0 \le \rho \le 1} \left[\frac{(1 - \mathrm{e}^{-2r(\rho)})(1-\rho)}{2\rho} \right] \tag{19}$$

where R is the code rate, $r(\rho)$ is the asymptotic spectral shape function, E_b/N_0 denotes the signal-to-noise ratio (SNR), and $(E_b/N_0)_{\mathrm{ML,threshold}}$ is the ML decoding threshold. We computed the upper bound on the ML decoding threshold in (19) numerically for the binary image of both WNRAA and WNRAAA code ensembles for several values of n and q. The results are given in Tables VI and VII, respectively. For comparison purposes, we also report in the table the binary-input AWGN Shannon limit. All codes perform within 0.05 dB from capacity.

B. Convergence Thresholds Under Iterative Decoding

In Table VIII, we report the iterative convergence thresholds for the binary image of WNRAA code ensembles on the AWGN channel for repeat factor $n = 3$ and $n = 5$. Unfortunately, while the ML thresholds improve with q and

TABLE VI
UPPER BOUNDS ON THE ML DECODING THRESHOLD OF THE BINARY IMAGE OF WNRAA CODES BASED ON DIVSALAR'S BOUND IN [19].

	$q=2$	$q=4$	$q=8$	$q=16$	$q=32$	Capacity
$n=3$	-0.437	-0.449 dB	-0.453 dB	-0.453 dB	-0.453 dB	-0.495 dB
$n=5$	-0.952	-0.953 dB	-0953 dB	-0.953 dB	-0.953 dB	-0.964 dB
$n=10$	-1.284	-1.284 dB	-1.284 dB	-1.284 dB	-1.284 dB	-1.286 dB

TABLE VII
UPPER BOUNDS ON THE ML DECODING THRESHOLD OF THE BINARY IMAGE OF WNRAAA CODES BASED ON DIVSALAR'S BOUND IN [19].

	$q=2$	$q=4$	$q=8$	$q=16$	$q=32$	Capacity
$n=3$	-0.453	-0.453 dB	-0.453 dB	-0.453 dB	-0.453 dB	-0.495 dB
$n=5$	-0.953	-0.953 dB	-0953 dB	-0.953 dB	-0.953 dB	-0.964 dB
$n=10$	-1.284	-1.284 dB	-1.284 dB	-1.284 dB	-1.284 dB	-1.286 dB

get closer to the capacity, the iterative convergence thresholds get worse with increasing values of q. We remark that if we remove the nonbinary weighters, the iterative decoding thresholds will improve, and they will be slightly better than those of binary RMA codes with the same repetition factor n.

However, for the QSC, the iterative decoding thresholds will be the same with and without the nonbinary weighter (random or fixed). This can be explained by the following argument. Let the symbol-wise log-likelihood ratio (LLR) for the ith received symbol r_i, $i=1,\dots,N$, be defined as the length-$(q-1)$ vector

$$\left(\ln\left(\frac{P(r_i|1)}{P(r_i|0)}\right),\dots,\ln\left(\frac{P(r_i|q-1)}{P(r_i|0)}\right)\right)$$

where $P(r_i|x_i)$ is the probability of receiving r_i when x_i is transmitted over the QSC. Under the assumption that we transmit the all-zero codeword, the symbol-wise LLR vectors will be of the form

$$\left(\ln\left(\frac{p}{(1-p)(q-1)}\right),\dots,\ln\left(\frac{p}{(1-p)(q-1)}\right)\right)$$

when we receive a zero (with probability $1-p$), or

$$\left(\overbrace{0,\dots,0}^{j-1},\ln\left(\frac{(1-p)(q-1)}{p}\right),\overbrace{0,\dots,0}^{q-1-j}\right) \tag{20}$$

when $r_i = j$, $j>0$. The probability of having such an LLR vector is $p/(q-1)$, independent of j (and of i). Since the nonbinary weighter will map a zero to a zero, the probability distribution of the LLR vector is *preserved* by the nonbinary weighter. Furthermore, due to properties of the rate-1 nonbinary accumulator (with no trellis termination), all symbol-wise LLR vectors of the form in (20) have the same probability, independent of j, even at the input of the Lth nonbinary accumulator (after decoding on the Lth nonbinary accumulator trellis). Due to this property, the nonbinary weighter at stage $L-1$ (after C_{L-1}) will also *preserve* the probability distribution of the symbol-wise LLR vector. The result follows by induction on the number of accumulators in the WNRMA code ensemble.

Finally, we remark that the above argument does not apply to the binary image of WNRMA code ensembles transmitted over the binary-input AWGN channel, since the *symmetry* of the QSC is lost.

TABLE VIII
CONVERGENCE THRESHOLDS FOR THE BINARY IMAGE OF WNRAA CODE ENSEMBLES FOR DIFFERENT VALUES OF THE REPETITION FACTOR n AND THE FIELD SIZE q ON THE AWGN CHANNEL.

	$q=2$	$q=4$	$q=8$	$q=16$	$q=32$
$n=3$	1.68 dB	1.94 dB	2.23 dB	2.45 dB	2.66 dB
$n=5$	2.77 dB	3.10 dB	3.46 dB	3.76 dB	4.07 dB

VI. CONCLUSION

In this paper, we analyzed the symbol-wise minimum distance properties of WNRMA code ensembles, where each encoder is followed by a nonbinary random weighter. We derived an exact closed-form expression for the IOWE of nonbinary accumulators. Based on that, we derived the ensemble-average WE of WNRMA code ensembles and analyzed its asymptotic behavior. Furthermore, we formally proved that the symbol-wise minimum distance of WNRMA code ensembles asymptotically grows linearly with the block length when $L\geq 3$ and $n\geq 2$, and $L=2$ and $n\geq 3$, for all powers of primes $q\geq 3$ considered. The asymptotic growth rate coefficient of the minimum distance of WNRAA and WNRAAA code ensembles for different values of the repetition factor n and the field size q were also computed. The asymptotic growth rates are very close to the GVB when q is large, but not too large. We also considered EXIT charts and analyzed the iterative convergence behavior of WNRMA code ensembles on the QSC. Finally, we considered the binary image of WNRMA code ensembles. We computed the asymptotic growth rates of their minimum distance and compared them with those of binary RMA code ensembles. It is shown that WNRMA code ensembles achieve higher $d_{\min}$ growth rates than their binary counterparts, and they get close to the GVB when q and n grow. Upper bounds on the ML decoding thresholds of the binary image of WNRMA code ensembles on the AWGN channel were also computed, and it was shown that WNRMA code ensembles perform very close to capacity under ML decoding. Unfortunately, iterative decoding is not able to fully exploit the performance of WNRMA code ensembles on the binary-input AWGN channel. In other words, WNRMA codes are excellent codes, but there is no efficient decoding algorithm to decode them close-to-ML. However, on the QSC, the iterative decoding performance is closer to capacity.

APPENDIX A

Numerical Evaluation of the Asymptotic Spectral Shape Function in (12)

The optimal value for the objective function in (11) can be either at a stationary point or at the boundary. To find the stationary points, we compute the partial derivatives with respect to $\beta_0,\beta_1,\dots,\beta_{L-1}$ and $\gamma_1,\dots,\gamma_L$. Setting the partial derivative with respect to β_0 equal to zero gives the equation

$$\left(\frac{(q-1)(1-\beta_0)}{\beta_0}\right)^{1/n-1}\cdot\left(\frac{\beta_1-\beta_0+\gamma_1}{\beta_0-2\gamma_1}\right)=\frac{1}{q-2}. \quad (21)$$

Setting the partial derivative with respect to γ_l, $l=1,\dots,L$, equal to zero results in the equation

$$\begin{aligned}&(q-2)^2\left(\beta_l-\beta_{l-1}+\gamma_l\right)\gamma_l^2\\ &\quad=(q-1)\left(\beta_{l-1}-2\gamma_l\right)^2\left(1-\beta_l-\gamma_l\right).\end{aligned} \quad (22)$$

Finally, setting the partial derivative with respect to β_l, $l=1,\dots,L-1$, equal to zero results in the equation

$$\frac{\beta_l^2(1-\beta_l-\gamma_l)(\beta_{l+1}-\beta_l+\gamma_{l+1})}{(1-\beta_l)^2(\beta_l-\beta_{l-1}+\gamma_l)(\beta_l-2\gamma_{l+1})}=\frac{q-1}{q-2}. \quad (23)$$

To determine a solution to the above set of equations, we choose the following strategy. First treat β_0 as a free parameter. Then, from (21), solve for β_1 as a (linear) function of γ_1 and insert the resulting expression into (22) for $l=1$. The resulting third order equation can now be solved for γ_1. Using (23) (with $l=1$), we can find β_2 as a (linear) function of γ_2 which we again can insert into (22) for $l=2$. The resulting third order equation can then be solved for γ_2. Continuing like this, we can find the remaining values for β_l and γ_l.

In general, we must also consider all combinations of boundary conditions in a systematic way, since the optimum value may be at the boundary. Details are omitted for brevity.

Appendix B
Proof of Theorem 4

The proof of Theorem 4 follows closely the proof of Theorem 6 in [8], which is inspired by the proof of Theorem 6 (or Theorem 9)[1] in [4] and the asymptotic techniques devised in [20]. We start by proving Lemma 3 stated below. The lemma is proved by induction on L.

Lemma 3: Let $\{h_N\}_{N\in\mathbb{N}}$ be a sequence of integers such that for any arbitrary $\eta>0$

$$\lim_{N\longrightarrow\infty}\frac{h_N}{N^\eta}=0 \text{ and } \lim_{N\longrightarrow\infty}\frac{\ln h_N}{h_N}=0.$$

Then,

$$\sum_{h=1}^{h_N}\bar{a}_h^{\mathcal{C}_{\text{WNRMA}}}=O\left(N^{1-\sum_{i=1}^{L}\left\lceil\frac{n}{2^i}\right\rceil+\eta}\right)$$

where L is the number of accumulators.

[1] In [4], Theorems 6 and 9 are the same result.

Proof: We prove the lemma by induction on the number of accumulators L. Consider first the case of $L=1$. We have

$$\begin{aligned}\sum_{h=1}^{h_N}\bar{a}_h^{\mathcal{C}_{\text{WNRA}}}&=\sum_{w=1}^{2h_N/n}\binom{N/n}{w}(q-1)^w\\ &\quad\times\sum_{h=1}^{h_N}\frac{\sum_{k=1}^{\lfloor\frac{nw}{2}\rfloor}\binom{N-h}{k}\binom{h-1}{k-1}\binom{h-k}{nw-2k}(q-1)^{k-nw}}{\binom{N}{nw}(q-2)^{2k-nw}}\\ &\quad\le\sum_{w=1}^{2h_N/n}N^{w-\lceil\frac{nw}{2}\rceil}g(w,N)\sum_{h=1}^{h_N}h^{nw+\lfloor\frac{nw}{2}\rfloor-3}\end{aligned}$$

where

$$\begin{aligned}g(w,N)&=\frac{(nw)!\mathrm{e}^{nw+w-1}\varphi_N(nw-1)}{n^w w^w}\\ &\quad\times\sum_{k=1}^{\lfloor\frac{nw}{2}\rfloor}\frac{(nw-2k)^{2k-nw}(q-1)^{k-nw+w}}{k^k(k-1)^{k-1}(q-2)^{2k-nw}}\end{aligned}$$

and we have used the Stirling's approximation $\binom{n}{k}\le\left(\frac{n\mathrm{e}}{k}\right)^k$ and the fact that $\prod_{i=0}^{l}(N-i)\ge\frac{N^{l+1}}{\phi_N(l)}$, with $\phi_\lambda(l)=\exp\left(\frac{l(l+1)}{2\lambda}\right)$. Also, note that the upper bound of $2h_N/n$ in the summation over w is due to the binomial $\binom{h-k}{nw-2k}$. In more detail, $h-k\ge nw-2k$, from which it follows that $nw\le h+k\le h+\lfloor nw/2\rfloor$, which implies that $w\le 2h/n$. Now, it follows that

$$\begin{aligned}\sum_{h=1}^{h_N}\bar{a}_h^{\mathcal{C}_{\text{RA}}}&\le\sum_{w=1}^{2h_N/n}N^{w-\lceil\frac{nw}{2}\rceil}g(w,N)h_N^{nw+\lfloor\frac{nw}{2}\rfloor-2}\\ &\le\frac{2h_N}{n}\max_{1\le w\le 2h_N/n}N^{w-\lceil\frac{nw}{2}\rceil}g(w,N)h_N^{nw+\lfloor\frac{nw}{2}\rfloor-2}\\ &\le\frac{2}{n}N^{1-\lceil\frac{n}{2}\rceil+\eta}g(1,N)h_N^{n+\lfloor\frac{n}{2}\rfloor-1}\\ &=O\left(N^{1-\lceil n/2\rceil+\eta}\right)\end{aligned}$$

for large enough N and for all $\eta>0$. Note that for large enough N, $N^{w-\lceil\frac{nw}{2}\rceil}$ dominates $g(w,N)h_N^{nw+\lfloor\frac{nw}{2}\rfloor-2}$, due to the conditions on h_N stated in the lemma. Now, assume that the statement of the lemma is true for the case of $L-1$. We get

$$\begin{aligned}&\sum_{h=1}^{h_N}\bar{a}_h^{\mathcal{C}_{\text{WNRMA}(L)}}\\ &\quad=\sum_{w=\lceil\frac{n}{2^{L-1}}\rceil}^{2h_N}\bar{a}_w^{\mathcal{C}_{\text{WNRMA}(L-1)}}\\ &\qquad\times\sum_{h=1}^{h_N}\sum_{k=1}^{\lfloor\frac{w}{2}\rfloor}\frac{\binom{N-h}{k}\binom{h-1}{k-1}\binom{h-k}{w-2k}(q-1)^{k-w}}{\binom{N}{w}(q-2)^{2k-w}}\\ &\quad\le\sum_{w=\lceil\frac{n}{2^{L-1}}\rceil}^{2h_N}\bar{a}_w^{\mathcal{C}_{\text{WNRMA}(L-1)}}N^{-\lceil\frac{w}{2}\rceil}g'(w,N)\sum_{h=1}^{h_N}h^{w+\lfloor\frac{w}{2}\rfloor-3}\end{aligned}$$

where

$$g'(w,N) = (w)!\mathrm{e}^{w-1}\varphi_N(w-1) \times \sum_{k=1}^{\lfloor \frac{w}{2} \rfloor} \frac{(w-2k)^{2k-w}(q-1)^{k-w}}{k^k(k-1)^{k-1}(q-2)^{2k-w}}$$

and $\mathcal{C}_{\mathrm{WNRMA}(l)}$ denotes the WNRMA code ensemble with l accumulators. Note that the lower bound of $\lceil n/2^{L-1} \rceil$ in the summation over w is due to the fact that the output size h of an accumulator is at least $\lceil w/2 \rceil$, where w is the input size. This is due to the binomial $\binom{h-k}{w-2k}$ and the upper bound of $\lfloor w/2 \rfloor$ in the summation over k. It follows that

$$\begin{aligned}
&\sum_{h=1}^{h_N} \bar{a}_h^{\mathcal{C}_{\mathrm{WNRMA}(L)}} \\
&\quad \leq \sum_{w=\lceil \frac{n}{2^{L-1}} \rceil}^{2h_N} \bar{a}_w^{\mathcal{C}_{\mathrm{WNRMA}(L-1)}} N^{-\lceil \frac{w}{2} \rceil} g'(w,N) h_N^{w+\lfloor \frac{w}{2} \rfloor - 2} \\
&\quad \leq O\left(N^{1-\sum_{i=1}^{L-1} \lceil \frac{n}{2^i} \rceil + \eta}\right) \\
&\qquad \times \max_{\lceil \frac{n}{2^{L-1}} \rceil \leq w \leq 2h_N} N^{-\lceil \frac{w}{2} \rceil} g'(w,N) h_N^{w+\lfloor \frac{w}{2} \rfloor - 2} \\
&\quad = O\left(N^{1-\sum_{i=1}^{L} \lceil \frac{n}{2^i} \rceil + \eta}\right)
\end{aligned}$$

for large enough N and for all $\eta > 0$. Above, we used the induction hypothesis in the second inequality. Also, note that for large enough N, $N^{-\lceil \frac{w}{2} \rceil}$ dominates $g'(w,N)h_N^{w+\lfloor \frac{w}{2} \rfloor - 2}$, due to the conditions on h_N stated in the lemma. ■

Lemma 4: Let $r^{\mathcal{C}_{\mathrm{WNRMA}}}(\rho;N)$ denote the Nth spectral shape function of the WNRMA code ensemble, defined as $r^{\mathcal{C}_{\mathrm{WNRMA}}}(\rho;N) = \frac{1}{N}\ln \bar{a}_{\lfloor \rho N \rfloor}^{\mathcal{C}_{\mathrm{WNRMA}}}$. Then,

$$r^{\mathcal{C}_{\mathrm{WNRMA}}}(\rho;N) \leq \frac{2L\ln(N+1)}{N} + r^{\mathcal{C}_{\mathrm{WNRMA}}}(\rho).$$

Proof: The proof of the lemma relies on the function $\psi(u,\rho)$, defined in (13). In particular, the proof of the lemma is by induction on L, following the same arguments as in the proof of Lemma 5 in [4], and is therefore omitted for brevity. ■

The final part of the proof of [4, Theorem 9] is also very general, and it can easily be extended to the case of WNRMA codes. In fact, the rest of the proof only relies on the following properties of $\psi(u,\rho)$.

1) $\psi(u,\rho)$ is continuous;
2) $\psi(u,\rho)$, for fixed u, is strictly increasing in $\rho < 1/2$;
3) $\frac{\psi(u,\rho)}{u}$, for fixed ρ, is decreasing in u; and
4) $\lim_{u \longrightarrow 0} \frac{\psi(u,\rho)}{u} < 0 \ \forall \rho < (q-1)/q$.

Finally, by using Lemmas 3 and 4 and the properties above, Theorem 4 is proved following the same arguments as in the proof of [4, Theorem 9]. Below, we will prove Properties 2 to 4. Property 1 follows from the definition of $\psi(u,\rho)$ given in (13).

A. Proof of Property 2

To prove Property 2, we compute the partial derivative of $\psi(u,\rho)$ with respect to ρ. Let the optimum value of γ (as a function of ρ) when solving the optimization problem in (13) be denoted by $\hat{\gamma}_\rho = \hat{\gamma}(\rho)$ and its derivative with respect to ρ as $\hat{\gamma}'_\rho = \hat{\gamma}'(\rho)$. Now,

$$\begin{aligned}
\frac{\partial \psi(u,\rho)}{\partial \rho} &= \mathbb{H}\left(\frac{\hat{\gamma}_\rho}{\rho}\right) + \frac{\rho\hat{\gamma}'_\rho - \hat{\gamma}_\rho}{\rho}\ln\left(\frac{\rho-\hat{\gamma}_\rho}{\hat{\gamma}_\rho}\right) \\
&\quad - \mathbb{H}\left(\frac{\hat{\gamma}_\rho}{1-\rho}\right) + \frac{\hat{\gamma}'_\rho(1-\rho)+\hat{\gamma}_\rho}{1-\rho}\ln\left(\frac{1-\rho-\hat{\gamma}_\rho}{\hat{\gamma}_\rho}\right) \\
&\quad + (1-\hat{\gamma}'_\rho)\mathbb{H}\left(\frac{u-2\hat{\gamma}_\rho}{\rho-\hat{\gamma}_\rho}\right) \\
&\quad + \frac{\hat{\gamma}'_\rho(u-2\rho)-u+2\hat{\gamma}_\rho}{\rho-\hat{\gamma}_\rho}\ln\left(\frac{\rho+\hat{\gamma}_\rho-u}{u-2\hat{\gamma}_\rho}\right) \\
&\quad + \hat{\gamma}'_\rho\ln(q-1) - 2\hat{\gamma}'_\rho\ln(q-2) \\
&= \mathbb{H}\left(\frac{\hat{\gamma}_\rho}{\rho}\right) - \frac{\hat{\gamma}_\rho}{\rho}\ln\left(\frac{\rho-\hat{\gamma}_\rho}{\hat{\gamma}_\rho}\right) - \mathbb{H}\left(\frac{\hat{\gamma}_\rho}{1-\rho}\right) \\
&\quad + \frac{\hat{\gamma}_\rho}{1-\rho}\ln\left(\frac{1-\rho-\hat{\gamma}_\rho}{\hat{\gamma}_\rho}\right) + \mathbb{H}\left(\frac{u-2\hat{\gamma}_\rho}{\rho-\hat{\gamma}_\rho}\right) \\
&\quad - \frac{u-2\hat{\gamma}_\rho}{\rho-\hat{\gamma}_\rho}\ln\left(\frac{\rho+\hat{\gamma}_\rho-u}{u-2\hat{\gamma}_\rho}\right) \\
&\quad + \hat{\gamma}'_\rho\left[\ln\left(\frac{\rho-\hat{\gamma}_\rho}{\hat{\gamma}_\rho}\right) + \ln\left(\frac{1-\rho-\hat{\gamma}_\rho}{\hat{\gamma}_\rho}\right)\right. \\
&\qquad - \mathbb{H}\left(\frac{u-2\hat{\gamma}_\rho}{\rho-\hat{\gamma}_\rho}\right) + \frac{u-2\rho}{\rho-\hat{\gamma}_\rho}\ln\left(\frac{\rho+\hat{\gamma}_\rho-u}{u-2\hat{\gamma}_\rho}\right) \\
&\qquad \left. + \ln(q-1) - 2\ln(q-2)\right].
\end{aligned} \tag{24}$$

Since $\tilde{\gamma}_\rho$ is a solution to the optimization problem in (13), it follows that $\tilde{\gamma}_\rho$ is a solution to the equation

$$\begin{aligned}
&\ln\left(\frac{\rho-\gamma}{\gamma}\right) + \ln\left(\frac{1-\rho-\gamma}{\gamma}\right) - \mathbb{H}\left(\frac{u-2\gamma}{\rho-\gamma}\right) \\
&+ \frac{u-2\rho}{\rho-\gamma}\ln\left(\frac{\rho+\gamma-u}{u-2\gamma}\right) + \ln(q-1) - 2\ln(q-2) = 0
\end{aligned} \tag{25}$$

which is obtained by taking the partial derivative with respect to γ of the objective function in (13) and setting it equal to zero. Substituting (25) (with $\gamma = \hat{\gamma}_\rho$) into (24), we get

$$\begin{aligned}
\frac{\partial \psi(u,\rho)}{\partial \rho} &= \mathbb{H}\left(\frac{\hat{\gamma}_\rho}{\rho}\right) - \frac{\hat{\gamma}_\rho}{\rho}\ln\left(\frac{\rho-\hat{\gamma}_\rho}{\hat{\gamma}_\rho}\right) - \mathbb{H}\left(\frac{\hat{\gamma}_\rho}{1-\rho}\right) \\
&\quad + \frac{\hat{\gamma}_\rho}{1-\rho}\ln\left(\frac{1-\rho-\hat{\gamma}_\rho}{\hat{\gamma}_\rho}\right) + \mathbb{H}\left(\frac{u-2\hat{\gamma}_\rho}{\rho-\hat{\gamma}_\rho}\right) \\
&\quad - \frac{u-2\hat{\gamma}_\rho}{\rho-\hat{\gamma}_\rho}\ln\left(\frac{\rho+\hat{\gamma}_\rho-u}{u-2\hat{\gamma}_\rho}\right) \\
&= \ln\left(\frac{\rho(1-\rho-\hat{\gamma}_\rho)}{(1-\rho)(\rho+\hat{\gamma}_\rho-u)}\right)
\end{aligned}$$

where the last equality follows from straightforward algebraic manipulations. Now, setting

$$\frac{\rho(1-\rho-\hat{\gamma}_\rho)}{(1-\rho)(\rho+\hat{\gamma}_\rho-u)} \leq 1 \tag{26}$$

gives $\rho \geq 1 - \hat{\gamma}_\rho/u \geq 1/2$, since $\hat{\gamma}_\rho \leq u/2$ (see (13)) and the denominator in (26) is nonnegative (since $\hat{\gamma}_\rho > u - \rho$, see (13)), from which it follows that, for fixed u, $\psi(u,\rho)$ is strictly increasing in ρ for $\rho < 1/2$.

B. Proof of Property 3

To prove Property 3, we compute the partial derivative of $\frac{\psi(u,\rho)}{u}$ with respect to u. We get

$$\frac{\partial}{\partial u}\left(\frac{\psi(u,\rho)}{u}\right)=\frac{1}{u^2}\left(u\frac{\partial\psi(u,\rho)}{\partial u}-\psi(u,\rho)\right).$$

Let the optimum value of γ (as a function of u) when solving the optimization problem in (13) be denoted by $\tilde{\gamma}_u=\tilde{\gamma}(u)$ and its derivative with respect to u as $\tilde{\gamma}_u'=\tilde{\gamma}'(u)$. Now,

$$\begin{aligned}\frac{\partial\psi(u,\rho)}{\partial u}&=-\ln\left(\frac{1-u}{u}\right)+\tilde{\gamma}_u'\ln\left(\frac{\rho-\tilde{\gamma}_u}{\tilde{\gamma}_u}\right)\\&\quad+\tilde{\gamma}_u'\ln\left(\frac{1-\rho-\tilde{\gamma}_u}{\tilde{\gamma}_u}\right)-\tilde{\gamma}_u'\mathbb{H}\left(\frac{u-2\tilde{\gamma}_u}{\rho-\tilde{\gamma}_u}\right)\\&\quad+\frac{\rho-\tilde{\gamma}_u-2\rho\tilde{\gamma}_u'+u\tilde{\gamma}_u'}{\rho-\tilde{\gamma}_u}\ln\left(\frac{\rho+\tilde{\gamma}_u-u}{u-2\tilde{\gamma}_u}\right)\\&\quad+(\tilde{\gamma}_u'-1)\ln(q-1)+(1-2\tilde{\gamma}_u')\ln(q-2)\\&=-\ln\left(\frac{1-u}{u}\right)+\ln\left(\frac{\rho+\tilde{\gamma}_u-u}{u-2\tilde{\gamma}_u}\right)\\&\quad-\ln(q-1)+\ln(q-2)\\&\quad+\tilde{\gamma}_u'\left[\ln\left(\frac{\rho-\tilde{\gamma}_u}{\tilde{\gamma}_u}\right)+\ln\left(\frac{1-\rho-\tilde{\gamma}_u}{\tilde{\gamma}_u}\right)\right.\\&\qquad-\mathbb{H}\left(\frac{u-2\tilde{\gamma}_u}{\rho-\tilde{\gamma}_u}\right)+\frac{u-2\rho}{\rho-\tilde{\gamma}_u}\ln\left(\frac{\rho+\tilde{\gamma}_u-u}{u-2\tilde{\gamma}_u}\right)\\&\qquad\left.+\ln(q-1)-2\ln(q-2)\right].\end{aligned}\tag{27}$$

Since $\tilde{\gamma}_u$ is a solution to the optimization problem in (13), it follows that $\tilde{\gamma}_u$ is a solution to (25). Substituting (25) (with $\gamma=\tilde{\gamma}_u$) into (27), we get

$$\begin{aligned}\frac{\partial\psi(u,\rho)}{\partial u}&=-\ln\left(\frac{1-u}{u}\right)+\ln\left(\frac{\rho+\tilde{\gamma}_u-u}{u-2\tilde{\gamma}_u}\right)\\&\quad-\ln(q-1)+\ln(q-2)\end{aligned}\tag{28}$$

from which it follows that

$$\begin{aligned}\frac{\partial}{\partial u}\left(\frac{\psi(u,\rho)}{u}\right)&=\frac{1}{u^2}\left(-\ln(1-u)+2\tilde{\gamma}_u\ln(\tilde{\gamma}_u)-\rho\ln(\rho)\right.\\&\quad-(1-\rho)\ln(1-\rho)\\&\quad+(1-\rho-\tilde{\gamma}_u)\ln(1-\rho-\tilde{\gamma}_u)\\&\quad-2\tilde{\gamma}_u\ln(u-2\tilde{\gamma}_u)\\&\quad+(\rho+\tilde{\gamma}_u)\ln(\rho+\tilde{\gamma}_u-u)\\&\quad\left.-\tilde{\gamma}_u(\ln(q-1)-2\ln(q-2))\right).\end{aligned}\tag{29}$$

Note that the equation in (25) can be simplified to

$$\ln\left(\frac{\gamma^2}{(u-2\gamma)^2}\right)=\ln\left(\frac{(q-1)(1-\rho-\gamma)}{(q-2)^2(\rho+\gamma-u)}\right).\tag{30}$$

Substituting (30) into (29), we get

$$\begin{aligned}\frac{\partial}{\partial u}\left(\frac{\psi(u,\rho)}{u}\right)&=\frac{1}{u^2}\left(-\ln(1-u)-\rho\ln(\rho)\right.\\&\quad-(1-\rho)\ln(1-\rho)\\&\quad+(1-\rho)\ln(1-\rho-\tilde{\gamma}_u)\\&\quad\left.+\rho\ln(\rho+\tilde{\gamma}_u-u)\right)\\&=\frac{1}{u^2}\left(-\ln(1-u)+\rho\ln\left(\frac{\rho+\tilde{\gamma}_u-u}{\rho}\right)\right.\\&\quad\left.+(1-\rho)\ln\left(\frac{1-\rho-\tilde{\gamma}_u}{1-\rho}\right)\right).\end{aligned}\tag{31}$$

The function $-\ln(x)$ is convex, and Jensen's inequality gives

$$\begin{aligned}-\ln(1-u)&=-\ln\left(\rho\frac{\rho+\tilde{\gamma}_u-u}{\rho}+(1-\rho)\frac{1-\rho-\tilde{\gamma}_u}{1-\rho}\right)\\&\leq-\rho\ln\left(\frac{\rho+\tilde{\gamma}_u-u}{\rho}\right)\\&\quad-(1-\rho)\ln\left(\frac{1-\rho-\tilde{\gamma}_u}{1-\rho}\right)\end{aligned}\tag{32}$$

from which it follows (by substituting the upper bound from (32) into (31)) that $\frac{\partial}{\partial u}\left(\frac{\psi(u,\rho)}{u}\right)\leq 0$, and the result follows.

C. Proof of Property 4

Calculating the partial derivative of the objective function in (13) with respect to γ and setting it equal to zero results in the equation in (30), which can be simplified to

$$\begin{aligned}&-q^2\gamma^3+(-4+uq^2-\rho q^2+4q)\gamma^2\\&+(-4uq-u^2q+u^2+4u-4\rho u+4\rho uq)\gamma\\&-u^2-\rho u^2q+\rho u^2+u^2q=0.\end{aligned}\tag{33}$$

Setting $u=0$ in (33), gives

$$-q^2\gamma^3+(-4-\rho q^2+4q)\gamma^2=0$$

with solutions

$$\gamma=\begin{cases}0,\\0,\\\frac{-4-\rho q^2+4q}{q^2}.\end{cases}$$

Since $\tilde{\gamma}_u$ is upper-bounded by $u/2$ and lower-bounded by 0 (see (13)), it follows that $\tilde{\gamma}_0=\tilde{\gamma}(u=0)=0$. Now, the limit

$$\begin{aligned}\lim_{u\longrightarrow 0}\frac{\psi(u,\rho)}{u}&=\lim_{u\longrightarrow 0}\frac{\partial\psi(u,\rho)}{\partial u}\\&=\ln\left(\frac{q-2}{q-1}\right)+\lim_{u\longrightarrow 0}\ln\left(\frac{(\rho+\tilde{\gamma}_u-u)\,u}{(u-2\tilde{\gamma}_u)\,(1-u)}\right)\\&=\ln\left(\frac{q-2}{q-1}\right)\\&\quad+\lim_{u\longrightarrow 0}\ln\left(\frac{\rho-2u+\tilde{\gamma}_u+\tilde{\gamma}_u'u}{1-2\tilde{\gamma}_u'-2u+2\tilde{\gamma}_u+2\tilde{\gamma}_u'u}\right)\\&=\ln\left(\frac{q-2}{q-1}\right)+\ln\left(\frac{\rho}{1-2\tilde{\gamma}_0'}\right)\\&=\ln\left(\frac{(q-2)\rho}{(q-1)(1-2\tilde{\gamma}_0')}\right)\end{aligned}\tag{34}$$

where the first and third equalities follow from l'Hôpital's rule, the second equality follows from (28), and the fourth equality follows under the assumption that $1-2\tilde{\gamma}_0'$ is nonzero. We will now show that this is indeed the case.

Since $\tilde{\gamma}_0 = 0$, we may write $\tilde{\gamma}_u = \tilde{\gamma}_0' u + O(u^2)$ (Taylor series expansion around $u = 0$). Substituting $\tilde{\gamma}_0' u + O(u^2)$ for γ in (33) and taking the limit as u approaches zero, we get

$$\begin{aligned}\left(-4-\rho q^2+4q\right)(\tilde{\gamma}_0')^2 + 4\,(q-1)\,(\rho-1)\,\tilde{\gamma}_0' \\ -1-\rho q+\rho+q = 0\end{aligned} \tag{35}$$

which has the solutions (when $-4-\rho q^2+4q$ is assumed to be nonzero)

$$\tilde{\gamma}_0' = \frac{-2(1-\rho)(q-1) \pm (q-2)\sqrt{\rho(1-\rho)(q-1)}}{4+\rho q^2-4q}$$

from which it follows that

$$1-2\tilde{\gamma}_0' = \frac{\rho(q-2)^2 \pm 2(q-2)\sqrt{\rho(1-\rho)(q-1)}}{4+\rho q^2-4q}. \tag{36}$$

Now, inserting this expression into (34), we get

$$\begin{aligned}&\lim_{u\longrightarrow 0}\frac{\psi(u,\rho)}{u} \\ &= \ln\left(\frac{\rho(4+\rho q^2-4q)}{(q-1)\left(\rho(q-2)\pm 2\sqrt{\rho(1-\rho)(q-1)}\right)}\right).\end{aligned} \tag{37}$$

Setting

$$\frac{\rho(4+\rho q^2-4q)}{(q-1)\left(\rho(q-2)\pm 2\sqrt{\rho(1-\rho)(q-1)}\right)} = 1$$

results in the equation

$$\rho\left(-(q-1)(q+2)+\rho q^2\right) = \pm 2(q-1)\sqrt{\rho(1-\rho)(q-1)}. \tag{38}$$

Squaring both sides of the equality in (38), we get (after some re-arrangement)

$$\begin{aligned}&q^4\rho^3 - 2q^2(q+2)(q-1)\rho^2 \\ &+\left((q-1)^2(q+2)^2+4(q-1)^3\right)\rho - 4(q-1)^3 = 0\end{aligned} \tag{39}$$

with solutions

$$\rho = \begin{cases}(q-1)/q, \\ (q-1)/q, \\ 4(q-1)/q^2.\end{cases}$$

The third solution is not valid, since we have assumed $-4-\rho q^2+4q$ to be nonzero. Thus, (39) has only a single solution. To prove that the limit in (37) is strictly less than zero for $\rho < (q-1)/q$ when $-4-\rho q^2+4q$ is nonzero, it is sufficient to evaluate the limit for some $\rho < (q-1)/q$ such that $-4-\rho q^2+4q$ is nonzero. In this respect, we choose $\rho = (q-1)/q^2$, which is strictly smaller than $\min\left((q-1)/q, 4(q-1)/q^2\right)$. Since the denominator in (36) is negative for this value of ρ, it follows from (36) (the $\pm$ will be a minus) that

$$\begin{aligned}\frac{(q-2)\rho}{(q-1)(1-2\tilde{\gamma}_0')} &= \frac{\rho(4+\rho q^2-4q)}{(q-1)(\rho(q-2)-2\sqrt{\rho(1-\rho)(q-1)})} \\ &= \frac{3}{2\sqrt{q^2-q+1}-(q-2)} \\ &< \frac{3}{2\sqrt{(q-1)^2}-(q-2)} = \frac{3}{q}\end{aligned}$$

where the strict inequality follows from the fact that $q^2-q+1 > q^2-2q+1 = (q-1)^2$. Now, since $q \geq 3$,

$$\lim_{u\longrightarrow 0}\frac{\psi(u,\rho)}{u} = \ln\left(\frac{(q-2)\rho}{(q-1)(1-2\tilde{\gamma}_0')}\right) < 0$$

for $\rho = (q-1)/q^2$.

Finally, when $-4-\rho q^2+4q$ is zero, (35) reduces to

$$4\,(q-1)\,(\rho-1)\,\tilde{\gamma}_0' - 1 - \rho q + \rho + q = 0$$

with solution $1-2\tilde{\gamma}_0' = 1/2$, from which it follows from (34) that

$$\lim_{u\longrightarrow 0}\frac{\psi(u,\rho)}{u} = \ln\left(\frac{8(q-2)}{q^2}\right) < 0$$

for $q \geq 5$. Note that for $q = 3$ and 4, $4(q-1)/q^2 \geq (q-1)/q$, and we can conclude that Property 4 is proved.

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